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A SOLUTION FOR SATELLITE ORBIT TIME IN UMBRA AND PENUMBRA WITH APPLICATION TO A LUNAR SATELLITE MISSION ANALYSIS

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GREENBELT, MARYLAND

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DYNAMICS AND ADVANCED MISSIONS

REPORT DR 105

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DELTA PROJECT OFFICE

NASA/GSFC

GREENBELT, MARYLAND

SUMMARY

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Equations comprising a solution for the time spent by a satellite in umbra and penumbra are derived. Umbra and penumbra regions are assumed to be right circular cones, and shadowing bodies are assumed spherical. Pertinent constants are evaluated for shadowing by both the earth and the moon. The solution is applied in predicting shadow time distributions for the AIMP mission (establishment of a lunar satellite orbit).

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NOMENCLATURE

a	semimajor axis of ellipse
b	semiminor axis of ellipse
D	distance between sun and body casting the shadow
e	eccentricity of ellipse
E	eccentric anomaly
i	orbit inclination
l	mean anomaly
L	orbit lifetime
M	center of the moon
p	umbra half-cone angle
p'	penumbra half-cone angle
r	orbit radius
r_a	orbit radius at apocenter
r_p	orbit radius at pericenter
R_e	radius of the earth
R_m	radius of the moon
R_s	radius of the sun
S	center of the sun
t	time
t_s	shadow time per orbit period
t_p	time of perigee passage
x	rectangular position coordinates
y	
z	
δ	angle between the moon-sun line and the osculating lunar orbit plane
Θ	true anomaly

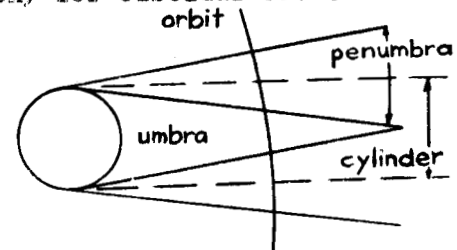
μ	gravitational constant
π	3.14159265
σ	angle in the orbit plane from the projection of the moon-sun line in the plane to any radius r
σ_p	value of σ at lunar orbit pericyynthion
τ	orbit period
ϕ	latitude
ω	argument of pericenter

A INTRODUCTION

In connection with the AIMP mission, a requirement was manifest to predict shadow time distributions for lunar satellite orbits. At the time of the study no lunar orbit shadow program was available to the Delta Project, and tabulated data available were based on solutions which assumed cylindrical shadow regions.

While cylindrical shadow theories are sufficiently accurate for most studies involving low or medium altitude satellite, accurate determination for high altitudes should be based on the conical model of umbra and penumbra regions.

The order of magnitude of the difference in the two solutions is indicated by Figure 1. This figure shows the maximum time in umbra and the maximum time in both umbra and penumbra (see sketch) for circular orbits of various radii. Times for both earth satellites and lunar satellites are shown. Times determined by a cylindrical shadow theory would lie between the "umbra only" and "umbra and penumbra" curves. The separation of these curves can be seen to be relatively large for high altitude orbits.



Because of error magnitudes possible in the injection into earth-to-moon transfer orbits, lunar orbits achieved in the AIMP mission may have an extremely wide range of eccentricities and altitudes. For this reason, a conical shadow theory was regarded as desirable.

A solution in which the umbra region is assumed conical, and the shadowing body is assumed spherical, is presented in Section B. Appendix C shows the applicability of this solution to the determination of time spent in the penumbra. The iterative solution is relatively simple and requires as inputs the size and shape of the satellite orbit and the angular position of the sun relative to the orbit plane.

Several authors have considered the shadow time determination problem (References 1 - 5). Most solutions assume the cylindrical shadow model. However, Fixler (Ref. 4) determines, by a different derivation, a solution which is reducible to that of Section B.

B. ANALYSIS

The geometry of the shadow time computation problem is shown in Figure 2. Here (A) is an edge view of the orbit plane viewed along a line perpendicular to the moon-sun line, and (B) is a projection showing the top view of the orbit plane. Scale is distorted, of course, for clarity. The lunar orbit plane intersects the shadow cone in a conic, generally an ellipse, hereafter called the "shadow ellipse"

If p is the umbra half-cone angle,

$$\sin p = \frac{R_{\odot} - R_{\oplus}}{MS} = 0.004646303.$$

The value of MS here is taken as the average value, 149.53×10^6 km (see Appendix A).

$$p = 0.26498$$

$$\psi_1 = 90^\circ - \delta - p$$

$$\psi_2 = 90^\circ - \delta + p$$

Then the semimajor axis of the shadow ellipse, a_s , can be obtained from

$$\frac{R_{\oplus}}{d} = \cos \psi_1 = \sin(\delta + p)$$

and

$$\frac{R_{\oplus}}{f} = \cos \psi_2 = \sin(\delta - p)$$

so that

$$a_s = \frac{d+f}{2} = \frac{R_{\oplus}}{2} \left[\frac{1}{\sin(\delta + p)} + \frac{1}{\sin(\delta - p)} \right]. \quad (1)$$

Note that since

$$\sin(\delta + p) \sin(\delta - p) = \sin^2 \delta - \sin^2 p, \quad (2)$$

$$a_s = R_{\oplus} \frac{\sin \delta \cos p}{\sin^2 \delta - \sin^2 p} \quad (3)$$

The shadow cone touches the moon along a minor circle of "latitude" p above the great circle normal to the moon-sun line. This minor circle touches the shadow ellipse in the orbit plane at two points, P and Q. The inclination of the lunar orbit plane to the normal great circle is $90^\circ - \delta$. Therefore, from spherical trigonometry

$$\sin c = \frac{\sin p}{\sin(90^\circ - \delta)} = \frac{\sin p}{\cos \delta}. \quad (4)$$

If (x', y') are coordinates in the lunar orbit plane with origin at the center of the shadow ellipse, the equation of the shadow ellipse is

$$\frac{x'^2}{a_s^2} + \frac{y'^2}{b_s^2} = 1 \quad (5)$$

Evaluating Eq (5) at the tangent points,

$$x' = a_s - d + R_e \sin c$$

$$y' = \pm R_e \cos c$$

gives

$$\frac{1}{b_s^2} = \frac{1}{R_e^2(1 - \sin^2 c)} \left[1 - \left(\frac{a_s - d}{a_s} + \frac{R_e}{a_s} \sin c \right)^2 \right] \quad (6)$$

Note that

$$a_s - d = \frac{d + f}{2} - d = \frac{f - d}{2} = \frac{R_e}{2} \left[\frac{1}{\sin(\delta - p)} - \frac{1}{\sin(\delta + p)} \right]$$

From Eq (2),

$$a_s - d = R_e \frac{\cos \delta \sin p}{\sin^2 \delta - \sin^2 p}$$

and

$$\frac{a_s - d}{a_s} = \frac{\tan p}{\tan \delta} \quad (7)$$

Also, $\frac{R_e}{a_s} \sin c = \frac{\tan p}{\sin \delta \cos \delta} (\sin^2 \delta - \sin^2 p)$

$$\begin{aligned} \frac{R_e}{a_s} \sin c + \frac{a_s - d}{a_s} &= \frac{\tan p}{\tan \delta} \left[1 + \frac{1}{\cos^2 \delta} (\sin^2 \delta - \sin^2 p) \right] \\ &= \frac{\sin p \cos p}{\sin \delta \cos \delta} \end{aligned} \quad (8)$$

$$R_e^2(1 - \sin^2 c) = R_e^2 \left(1 - \frac{\sin^2 p}{\cos^2 \delta} \right) \quad (9)$$

Substitution of Eqs (8) and (9) in (6) gives

$$\frac{1}{b_s^2} = \frac{1}{R_e^2} \left(1 - \frac{\sin^2 p}{\sin^2 \delta} \right) \quad (10)$$

The equation of the ellipse in moon-centered coordinates (x,y) is

$$\frac{(x + a_s - d)^2}{a_s^2} + \frac{y^2}{b_s^2} = 1$$

or, in polar coordinates,

$$\left(\frac{r \cos \sigma}{a_s} + \frac{a_s - d}{a_s} \right)^2 + \frac{r^2 \sin^2 \sigma}{b_s^2} = 1 \quad (11)$$

The equation for the lunar satellite orbit is

$$r = \frac{a(1 - e^2)}{1 + e \cos(\sigma - \sigma_p)} \quad (12)$$

Then, if values for δ and σ_p are specified at any time, a simultaneous solution of Eqs (11) and (12) gives the two values of σ at which the satellite orbit and the shadow ellipse intersect. Since the equations are transcendental, the solution must involve an iterative procedure. Eq (12) may be used to eliminate r in Eq (11).

However, Eq (11) can be simplified without resorting to expansions.

The equation of the form

$$\frac{(x+g)^2}{a_s^2} + \frac{y^2}{b_s^2} = 1$$

can be written in polar coordinates as

$$\frac{(r^2 \cos^2 \sigma + 2rg \cos \sigma + g^2)}{a_s^2} + \frac{r^2(1 - \cos^2 \sigma)}{b_s^2} = 1$$

or

$$r^2 \left[\frac{a_s^2}{b_s^2} + \cos^2 \sigma \left(1 - \frac{a_s^2}{b_s^2} \right) \right] + r [2g \cos \sigma] + g^2 - a_s^2 = 0. \quad (13)$$

The discriminate reduces to

$$B^2 - 4AC = 4a_s^2 \left[1 + \left(\frac{a_s^2 - g^2}{b_s^2} - 1 \right) \sin^2 \sigma \right]. \quad (14)$$

From

$$d = \frac{R_s}{\sin(\delta + p)},$$

$$q = a_s - d$$

and Eqs (3) and (10),

$$\frac{a_s^2 - g^2}{b_s^2} - 1 = \cot^2 \delta \quad (15)$$

$$\frac{a_s^2}{b_s^2} = \frac{\cos^2 p}{\sin^2 \delta - \sin^2 p} \quad (16)$$

$$1 - \frac{a_s^2}{b_s^2} = - \frac{\cos^2 \delta}{\sin^2 \delta - \sin^2 p} \quad (17)$$

Substitution of Eqs (15), (16) and (17) into Eq (13) gives the final equation of the shadow ellipse in polar, moon-centered coordinates.

$$r = R_s \frac{-\cos \sigma \cos \delta \sin p \pm \cos p \sqrt{1 - \cos^2 \delta \cos^2 \sigma}}{\cos^2 p - \cos^2 \delta \cos^2 \sigma} \quad (18)$$

The positive sign should be chosen in the numerator of Eq (18). That the negative sign is redundant can be seen most easily if the small angle p is set equal to zero. Then

$$r = \pm \left(\frac{1}{\sqrt{1 - \cos^2 \delta \cos^2 \sigma}} \right) \quad (p=0)$$

Since the radius vector is by definition a positive quantity, the negative sign is redundant.

Substitution for r from Eq (12) gives the final equation, to be solved for σ , the angular coordinate of the intersection of the orbit and the shadow ellipse.

$$F(\sigma) = -\frac{a(1-e^2)}{R_e}(\cos^2 p - \cos^2 \delta \cos^2 \sigma) + [1 + e \cos(\sigma - \sigma_p)](-\cos \sigma \cos \delta \sin p + \cos p \sqrt{1 - \cos^2 \delta \cos^2 \sigma}) = 0 \quad (19)$$

This transcendental equation must be solved by iteration. Newton's method gives

$$\sigma_{n+1} = \sigma_n - \frac{F(\sigma_n)}{F'(\sigma_n)} \quad (20)$$

where

$$F'(\sigma) = -2 \frac{a(1-e^2)}{R_e} \cos^2 \delta \sin \sigma \cos \sigma - e \sin(\sigma - \sigma_p)(-\cos \sigma \cos \delta \sin p) + \cos p \sqrt{1 - \cos^2 \delta \cos^2 \sigma} + [1 + e \cos(\sigma - \sigma_p)] \sin \sigma \cos \delta [\sin p + \frac{\cos p \cos \delta \cos \sigma}{\sqrt{1 - \cos^2 \delta \cos^2 \sigma}}]$$

Equations (19) and (20) then comprise the means of solution for the angular positions in orbit at which the orbit intersects the umbra cone. The time in shadow can then be determined easily from Kepler's equation,

$$l = E - e \sin E = \sqrt{\frac{\mu}{a^3}}(t - t_p) \quad (21)$$

where

$$\sin E = \frac{\sqrt{1-e^2} \sin \theta}{1 + e \cos \theta}$$

$$\cos E = \frac{e + \cos \theta}{1 + e \cos \theta}$$

$$\theta = \sigma - \sigma_p$$

Certain special cases have not been considered in the above solution. In Appendix B, the special case in which $\delta < p$ and the shadow conic intersection is a hyperbola is considered. The solution is shown to be the same as in the elliptical intersection case (Eq 18). Also, in Appendix C the solution for the penumbra intersection is shown to be similar to Eq. 18.

C. Application to Computation of Lunar Satellite Shadow Time

The computation routine of Section B was programmed in Fortran IV and incorporated in an orbit integration program in order to study shadow requirements of the AIMP lunar satellite mission. The integration program provides inputs (δ , σ_p , a , e as functions of time) to the shadow computation routine.

Orbits to be studied with the shadow routine were generated by the following procedure.

- (1) The nominal velocity magnitude, azimuth angle and flight path angle at third stage burnout (i.e. injection into the earth-to-moon transfer orbit) of the Delta DSV-3E vehicle were taken as means of three normal independent distributions of specified standard deviations.
- (2) Non-nominal sets of burnout conditions were generated from these three distributions for a large number of cases by means of a random number generating routine.
- (3) Each set of non-nominal burnout conditions was used as input to the Interplanetary Trajectory Encke Method (ITEM) program, which for each of the error conditions, as well as initial lunar orbit elements resulting from various fourth stage firing times.
- (4) For each transfer orbit one set of initial lunar orbit elements which appeared to represent a long lifetime lunar orbit was chosen as input to the lunar orbit integration program. If the first selection failed to achieve the desired six month lifetime, one or two other sets of elements were tested. The lunar orbit integration program used is a variation-of-parameters program written by Dr. William Kaula with modifications by the Douglas Aircraft Company and the Delta Project Office and incorporating the shadow routine described in Section B.
- (5) Shadow and lifetime data were tabulated for the lunar orbit having the longest lifetime of the cases investigated for each transfer orbit.

Results for a sample of one hundred transfer trajectories are shown in the following table. Here, r_p , r_a , a , e , i and ω are the initial lunar orbit elements (pericynthion and apocynthion radii in km, semimajor axis in moon radii, eccentricity, inclination and argument of pericynthion in degrees relative to the lunar equator and vernal equinox). Also, L is the lunar orbit lifetime in days, t_{smax} is the maximum shadow time experienced in any orbit period, \bar{t}_s is the average shadow time per orbit, and % t is the percentage of orbit time spent in shadow. The time at which the shadow time first exceeds 1, 1.5, 2, 2.5 and 3 hours is also given. Here "shadow" means umbra; no penumbra investigations were made for this mission.

Orbits having apocynthion radii greater than 25,000 nautical miles (46,300 km) were not investigated as being of questionable stability. The integration program used is not suitable for stability determination since accuracy degenerates for high altitude orbits. Also seven transfer orbits of the sample did not approach sufficiently close to the moon to achieve a lunar orbit. These cases are indicated in the table.

Figure 3 summarizes the shadow and lifetime data. For a given time after lunar orbit injection (abscissa scale) the ordinate scale indicates the number of cases having a longer lifetime and also having a maximum shadow time per orbit less than 1, 1.5, 2, 2.5, or 3 hours. The upper curve shows lifetime data without maximum shadow time requirements. Figure 4 shows histograms of the maximum and average shadow times per orbit.

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1. Cunningham, Fred G. "Calculation of the Eclipse Factor for Elliptical Satellite Orbits", NASA TN D-1347, June 1962
2. Wolverton, R. (editor) "Flight Performance Handbook for Orbital Operations" STL, September 1961
3. Patterson, G. "Graphical Method for Prediction of Time in Sunlight for a Circular Orbit", ARS Journal, Vol. 31, No. 3, p 441.
4. Sol Zalel Fixler, "Umbra and Penumbra Eclipse Factors for Satellite Orbits", TN 1455, Vol 2, AIAA J.
5. Stoddard L. G. "Eclipse of Artificial Earth Satellite", Astron. Sciences Rev. Vol III, pp 9-16.

APPENDIX A

Constants for Cone Angle Determination

Complexity of the shadow time computation can be greatly reduced by using mean values of geoid radius, R_e , and earth orbit radius, D , in the formulas for umbra and penumbra half cone angles (Eqs C.1, C.2). These quantities vary between the following limits.

$$6256.77 \leq R_e \leq 6378.165 \text{ km}$$

$$147.03 \times 10^6 \leq D \leq 152.03 \times 10^6 \text{ km}$$

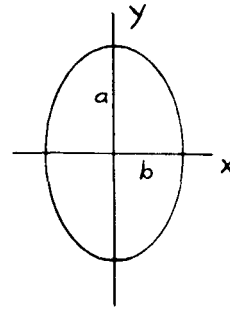
The mean values to be used in general programs may be computed in several different ways.

1. Mean Geoid Radii

- a. Radius of the sphere having volume equivalent to that of the oblate spheroid

The volume of an oblate spheroid (a volume generated by rotating an ellipse about its minor axis) is given by

$$V = 2\pi \int_0^b y^2 dx \quad (\text{A.1})$$



where the minor axis coincides with the x-axis. The equation of the ellipse in the coordinates shown in the sketch is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1. \quad (\text{A.2})$$

Substituting for y^2 in (A.1) gives

$$V = 2\pi \int_0^b a^2 \left(1 - \frac{x^2}{b^2}\right) dx = \frac{4}{3} \pi a^2 b$$

If R_v is the radius of the sphere of equivalent volume,

$$\frac{4}{3} \pi R_v^3 = \frac{4}{3} \pi a^2 b$$

$$R_v = \sqrt[3]{a^2 b} \quad (\text{A.3})$$

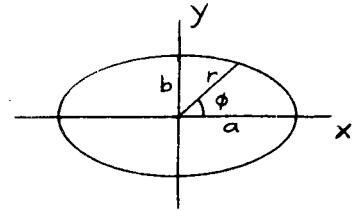
- b. Radius of the ellipse averaged over latitude angle

In the elliptical cross-section of an oblate spheroid, the radius may be expressed as a function of latitude as follows:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{r^2 \cos^2 \phi}{a^2} + \frac{r^2 \sin^2 \phi}{b^2} = 1$$

$$r^2 = \frac{1}{\frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2}} \quad (\text{A.4})$$



The radius averaged over latitude is

$$R_\phi = \frac{1}{\pi/2} \int_0^{\pi/2} r(\phi) d\phi, \quad (\text{A.5})$$

Substitution of (A.4) in (A.5) and simplification gives

$$R_\phi = \frac{2b}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \cos^2 \phi}} \quad (\text{A.6})$$

where $k^2 = 1 - \frac{b^2}{a^2} = e^2$

Then

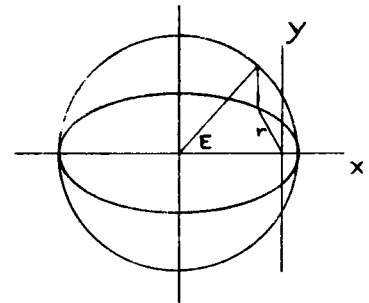
$$R_\phi = \frac{2b}{\pi} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \frac{2b}{\pi} F(k, \frac{\pi}{2}) \quad (\text{A.7})$$

where F is the elliptic integral of the first kind.
For $k^2 \ll 1$

$$R_\phi = b \left[1 + 2 \frac{k^2}{8} + 9 \left(\frac{k^2}{8} \right)^2 + 50 \left(\frac{k^2}{8} \right)^3 + \dots \right] \quad (\text{A.8})$$

- c. Radius of the circle having a circumference equivalent to that of the spheroid cross-section.

Determination of the circumference of an ellipse is facilitated by definition of an auxiliary variable, the eccentric anomaly E , and choice of coordinates centered at the focus of the elliptical cross-section. Then, if s is arc length along the circumference,



$$ds^2 = dx^2 + dy^2$$

$$x = a(\cos E - e) \quad dx = -a \sin E dE$$

$$y = b \sin E \quad dy = b \cos E dE$$

$$ds = a \sqrt{1 - e^2 \cos^2 E} dE$$

$$s = 2a \int_0^{\pi} \sqrt{1 - e^2 \cos^2 E} dE$$

$$s = 4aE\left(\frac{\pi}{2}, k\right), \quad k = e$$

If the circumference of the equivalent circle is $2\pi R_c$,

$$R_c = \frac{2a}{\pi} E\left(\frac{\pi}{2}, k\right) \quad (\text{A.9})$$

For $k^2 \ll 1$

$$R_c = a \left[1 - 2 \frac{k^2}{8} - 3 \left(\frac{k^2}{8} \right)^2 - 10 \left(\frac{k^2}{8} \right)^3 - \dots \right] \quad (\text{A.10})$$

- d. Radius of the circle having an area equivalent to that of the spheroid cross-section.

For the ellipse given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the area is

$$\begin{aligned} A &= 4 \int_0^a y dx = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}} dx \\ &= \frac{2b}{a} \left[x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right]_0^a \end{aligned}$$

$$A = \pi ab \quad (\text{A.11})$$

If the radius of the circle of equivalent area is R_a ,

$$R_a = \sqrt{ab} \quad (\text{A.12})$$

2. Mean Earth Orbit Radius

a. Astronomical Unit

The astronomical unit is the mean distance from the sun to a fictitious planet, the mass and period of which are those used by Gauss in his determination of the solar gravitation constant.

$$R = 149.53 \times 10^6 \text{ km}$$

b. Radius averaged over orbit central angle

$$\begin{aligned} r_o &= \frac{1}{\pi} \int_0^\pi r(\theta) d\theta = \frac{1}{\pi} \int_0^\pi \frac{a(1-e^2)}{1+e \cos \theta} d\theta \\ &= \frac{a(1-e^2)}{\pi} \left[\frac{2}{\sqrt{1-e^2}} \tan^{-1} \frac{\sqrt{1-e^2} \tan \frac{\theta}{2}}{1+e} \right]_0^\pi \\ r_o &= a \sqrt{1-e^2} \quad (\text{A.13}) \end{aligned}$$

c. Radius of the circle of area equivalent to that enclosed by the earth orbit

From (A.11)

$$\pi ab = \pi r_a^2$$

$$r_a = \sqrt{ab} = a(1-e^2)^{\frac{1}{2}} \quad (\text{A.14})$$

d. Time average of orbit radius

Kepler's equation is

$$l = E - e \sin E$$

where

$$l = \sqrt{\frac{\mu}{a^3}} (t - t_0)$$

Then

$$\frac{dE}{dl} = \frac{1}{1 - e \cos E}$$

$$r = a(1 - e \cos E)$$

$$\frac{dE}{dl} = \frac{a}{r}$$

$$\frac{dE}{dt} = \sqrt{\frac{\mu}{a^3}} \frac{dE}{dl} = \frac{1}{r} \sqrt{\frac{\mu}{a}}$$

The orbit radius averaged over time is

$$r_t = \frac{1}{\tau} \int_0^{\tau} r(t) dt$$

where τ is the orbit period.

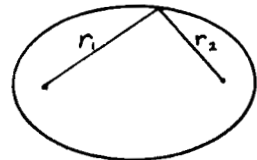
$$\begin{aligned} r_t &= \frac{1}{\tau} \int_0^{2\pi} r^2(E) \sqrt{\frac{a}{\mu}} dE \\ &= \frac{a}{2\pi} \int_0^{2\pi} (1 - e \cos E)^2 dE \\ r_t &= a \left(1 + \frac{e^2}{2} \right) \end{aligned} \quad (\text{A.15})$$

e. Radius of the circular orbit of equivalent period

$$r_c = a \quad (\text{A.16})$$

This is also the average obtained by considering the sum of the radii from each focus to every point on the ellipse.

$$r_1 + r_2 = 2a$$



f. Radius of the circle of equivalent circumference

From (A.10)

$$r_c = a \left[1 - 2 \frac{k^2}{8} - 3 \left(\frac{k^2}{8} \right)^2 - \dots \right] \quad (\text{A.17})$$

3. Numerical Evaluations

Mean Geoid Radius

Type of Average	Formula	Value, km
Sphere of equivalent volume	$R_v = \sqrt{a^2 b}$	6371.02
Radius averaged over latitude	$R_\phi = \frac{2b}{\pi} F(k, \frac{\pi}{2})$	6367.45
Circle of equivalent circumference	$R_c = \frac{2a}{\pi} E(k, \frac{\pi}{2})$	6367.47
Circle of equivalent area	$R_a = \sqrt{ab}$	6367.46

Assumed constants:

$$a = 6378.16 \text{ km}$$

$$b = 6356.77 \text{ km}$$

$$e = 0.0818292$$

Mean Orbit Radius

Type of Average	Formula	Value, r/a
Astronomical unit	Defined, $1.4953 \times 10^8 \text{ km}$	
Radius averaged over central angle	$r_\theta = a \sqrt{1 - e^2}$	0.9998601
Circle of equivalent area	$r_a = \sqrt{ab} = a(1 - e^2)^{\frac{1}{4}}$	0.9999300
Radius averaged over time	$r_t = a \left(1 + \frac{e^2}{2} \right)$	1.0001399
Equivalent period	$r_T = a$	1.0
Equivalent circumference	$r_c = \frac{2a}{\pi} E(k, \frac{\pi}{2})$	0.99993006

Assumed earth orbit eccentricity $e = 0.0167268$

4. Values for Cone Angle Calculations

The tables of the previous section indicate that three of the four geoid radius averages differ by the order of magnitude of the errors in the equatorial and polar radii

$$a = 6378.16 \pm 0.02 \text{ km}$$

$$b = 6356.77 \pm 0.05 \text{ km}$$

Therefore based on these values of the physical constants, a suitable value for the average geoid radius is 6367.46 km.

Also, the tables show that all computed values of mean earth orbit radii lie within the range of uncertainty in the astronomical unit.

$$\text{a.u.} = 149.53 \times 10^6 \pm 0.03 \times 10^6 \text{ km,}$$

which is, therefore, a suitable mean orbit radius.

For problems requiring greater accuracy than can be achieved with mean cone angle values, ephemerides may be used to compute exact values for the time in question. Figure A-1 shows the variations in umbra and penumbra half-cone angles during 1966 for the earth and the moon.

APPENDIX B

Special Case $\delta < p$

For the case $\delta < p$ the conic section formed by the intersection of the orbit plane and the shadow cone is a hyperbola. However, the equation of this shadow hyperbola is the same as that of the ellipse in the more general case $\delta > p$ (Eq 18). Verification of Eq 18 in the hyperbolic case may be carried out in a scheme similar to that of Section B. The geometry is shown in Figure B-1.

As before, from Figure B-1 (A),

$$d = \frac{R_e}{\sin(p + \delta)}, \quad (\text{B.1})$$

and, from (B)

$$\sin c = \frac{\sin p}{\cos \delta} \quad (\text{B.2})$$

The equation of the hyperbola in moon-centered coordinates is

$$\frac{(x - g')^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\text{B.3})$$

$$\begin{aligned} \text{where } b^2 &= a^2(e^2 - 1) \\ g' &= a + d \end{aligned}$$

The shadow cone is tangent to the moon along a minor circle of latitude p above the great circle normal to the M-S line, as in the ellipse case. Then any curve lying in the shadow cone (and, in particular, the shadow hyperbola) is also tangent to the moon on this circle at

$$x_1 = R_e \sin c = R_e \frac{\sin p}{\cos \delta} \quad (\text{B.4})$$

$$y_1 = R_e \cos c$$

in moon-centered coordinates. Differentiating (B.3) gives

$$\frac{2(x - g')}{a^2} - \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\text{or } \frac{dy}{dx} = \frac{b^2}{a^2} \frac{x - g'}{y} \quad (\text{B.5})$$

At (x_1, y_1)

$$\begin{aligned} \left. \frac{dy}{dx} \right|_1 &= \frac{R_e \cos c}{R_e \frac{\sin p}{\cos \delta} - R_e \frac{\cos \delta}{\sin p}} \\ &= \frac{\cos c}{\sin c - \frac{1}{\sin c}} = -\tan c \end{aligned}$$

$$\left. \frac{dy}{dx} \right|_1 = \frac{b^2}{a^2} \left[\frac{R_e \sin c - g'}{R_e \cos c} \right] \quad (\text{B.6})$$

From (B.6) and $g' = a + d$

$$\frac{a^2}{b^2} = \frac{a+d}{R_e \sin c} - 1 \quad (\text{B.7})$$

Substituting (B.4) in (B.3) gives

$$\frac{a^2}{b^2} + \frac{a^2}{R_e^2 \cos^2 c} = \frac{(R_e \sin c - g')^2}{R_e^2 \cos^2 c}. \quad (\text{B.8})$$

Substituting (B.7) in (B.8) and simplifying yields

$$a = \frac{dR_e(\sin^2 c + 1) - (R_e^2 + d^2)\sin c}{-R_e(1 + \sin^2 c) + 2d \sin c},$$

which in turn, on substitution of (B.1) and (B.2) gives

$$a = R_e \frac{\cos p \sin \delta}{\sin^2 p - \sin^2 \delta}. \quad (\text{B.9})$$

Except for sign, this expression is the same as Eq (3) (part B).

Also, from (B.7)

$$\frac{1}{b^2} = \frac{a+d - R_e \sin c}{a^2 R_e \sin c}. \quad (\text{B.10})$$

From (B.9), (B.1) and (B.2)

$$a + d - R_e \sin c = R_e \left[\frac{\sin p \cos \delta}{\sin^2 p - \sin^2 \delta} - \frac{\sin p}{\cos \delta} \right]$$

so that (B.10) can be reduced to

$$\frac{1}{b^2} = \frac{1}{R_e^2} \left(\frac{\sin^2 p}{\sin^2 \delta} - 1 \right). \quad (\text{B.11})$$

The equation of the hyperbola may be derived as follows. For $x = r \cos \sigma$ and $y = r \sin \sigma$, (B.3) gives

$$r^2 \left(\frac{\cos^2 \sigma}{a^2} - \frac{1 - \cos^2 \sigma}{b^2} \right) - \frac{2(a+d)}{a^2} \cos \sigma r + \frac{(a+d)^2}{a^2} - 1 = 0$$

In the usual quadratic notation, designate

$$\begin{aligned} A &= \frac{\cos^2 \sigma}{a^2} - \frac{1 - \cos^2 \sigma}{b^2} \\ B &= \frac{-2(a+d)}{a^2} \cos \sigma \\ C &= \frac{(a+d)^2}{a^2} - 1 \end{aligned} \quad (\text{B.12})$$

Since, from (B.9) and (B.11),

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{R_e^2} \frac{(\sin^2 p - \sin^2 \delta)}{\sin^2 \delta \cos^2 p} \cos^2 \delta$$

so that

$$A = \frac{1}{R_e^2} \frac{(\sin^2 p - \sin^2 \delta)}{\sin^2 \delta} \left(\frac{\cos^2 \delta \cos^2 \sigma}{\cos^2 p} - 1 \right)$$

Also,

$$B^2 - 4AC = \frac{4}{a^2} \left[1 + \sin^2 \sigma \left(\frac{2ad + d^2}{b^2} - 1 \right) \right]$$

Then

$$\begin{aligned} r &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \\ &= \frac{(a+d) \cos \sigma \pm a \sqrt{1 + \sin^2 \sigma \left(\frac{2ad + d^2}{b^2} - 1 \right)}}{a^2 \left[\left(\frac{1}{b^2} + \frac{1}{a^2} \right) \cos^2 \sigma - \frac{1}{b^2} \right]} \end{aligned} \quad (B.13)$$

From (B.1), (B.9) and (B.10)

$$\frac{2ad + d^2}{b^2} - 1 = \cot^2 \delta \quad (B.14)$$

and

$$a + d = \frac{R_e \sin p \cos \delta}{\sin(p + \delta) \sin(p - \delta)} \quad (B.15)$$

Substitution of (B.14) and (B.15) in (B.13) yields

$$r = R_e \frac{-\sin p \cos \delta \cos \sigma + \cos p \sqrt{1 - \cos^2 \delta \cos^2 \sigma}}{\cos^2 p - \cos^2 \delta \cos^2 \sigma} \quad (B.16)$$

This equation is exactly the same as Eq (18), Section B.

APPENDIX C

Penumbra Determination

The analysis of Section B is applicable to determination of time spent in penumbra as well as time spent in umbra. Figure C-1 shows the geometry of the penumbra problem. Here (A) shows the penumbra and umbra shadow-cones, with their half-cone angles p^1 , and p , respectively.

$$\sin p = \frac{R_o - R_e}{D} \quad (C.1)$$

$$\sin p' = \frac{R_o + R_e}{D} \quad (C.2)$$

Values for the earth and the moon are given in the following table

	p , deg	p^1 , deg
earth	0.2644 <u>366</u>	0.2693 <u>218</u>
moon	0.2662 <u>145</u>	0.2675 <u>467</u>

The geometry of the penumbra problem is obviously the same as that of the umbra problem. The magnitude of the half-cone angle changes from p to p^1 , and the vertex of the umbra cone is on the opposite side of the central body from the vertex of the penumbra cone. In fact, the analysis of Section B holds in the penumbra case if the transformation

$$p \rightarrow -p^1$$

is made.

This solution can be verified from Figure C-1.

From (B)

$$d = \frac{R_e}{\sin(\delta - p')} \quad (C.3)$$

$$f = \frac{R_e}{\sin(\delta + p')} \quad (C.4)$$

Then

$$a_s = \frac{d+f}{2} = \frac{R_e}{2} \left[\frac{1}{\sin(\delta - p')} + \frac{1}{\sin(\delta + p')} \right] \quad (C.5)$$

From (C)

$$\frac{x_1'^2}{a^2} + \frac{y_1'^2}{b^2} = 1$$

$$x'_1 = -a + f - R_e \sin c = a - d - R_e \sin c$$

$$y'_1 = \pm R_e \cos c$$

Then

$$\begin{aligned} \frac{1}{b_s^2} &= \frac{1}{y'^2} \left(1 - \frac{x'^2}{a^2} \right) \\ &= \frac{1}{R_e^2 (1 - \sin^2 c)} \left[1 - \left(\frac{a_s - d}{a_s} - \frac{R_e}{a_s} \sin c \right)^2 \right] \end{aligned} \quad (C.6)$$

Also

$$a_s - d = \frac{d+f}{2} - d = \frac{R_e}{2} \left[\frac{1}{\sin(\delta+p')} - \frac{1}{\sin(\delta-p')} \right] \quad (C.7)$$

The substitution $p' \rightarrow p$ can be seen to transform (C.1), (C.2), (C.3), (C.4), (C.5), (C.6) and (C.7) into the corresponding equations of the umbra problem (Section B, Eqs (1), (6)).

The equation of the shadow ellipse in moon coordinates is

$$\frac{(x - a_s + f)^2}{a_s^2} + \frac{y^2}{b_s^2} = 1.$$

However,

$$-a + f = -a + 2a - d = a - d$$

so that the ellipse equation becomes

$$\frac{(x + a_s - d)^2}{a_s^2} + \frac{y^2}{b_s^2} = 1 \quad (C.8)$$

which is the same as the equation preceding Eq (11), Section B.

RUN	r _p	r _a	a	e	i	ω	L	t _{max}	T _s	T _t	TIME (DAYS) WHEN SHADOW EXCEEDS				
											1h	1.5h	2h	2.5h	3h
1	NO	ORBITS													
2	6277	7619	4.573	.2103	172.13	129.15	180	1.493	1.307	7.40	0	—	—	—	—
3	5769	10706	4.740	.2997	174.05	168.49	180	1.677	1.348	7.23	0	28	—	—	—
4	7973	34917	12.339	.6282	158.43	179.61	180	0.673	0.130	.17	—	—	—	—	—
5	4434	15143	5.632	.5470	175.65	309.53	180	2.520	1.192	4.94	0	0	20	—	—
6	3197	4567	2.234	.1764	162.09	196.39	180	.942	.734	12.17	—	—	—	—	—
7	3469	44902	13.916	.8566	149.48	170.35	19.5	.528	.426	.45	—	—	—	—	—
8	NO	ORBITS													
9	4683	11846	4.755	.4334	113.33	129.15	180	1.282	.159	.85	84	—	—	—	—
10	5845	9815	4.505	.2535	174.65	246.50	180	1.413	1.141	.07	0	—	—	—	—
11	3726	23386	7.800	.7252	164.85	172.72	180	4.276	1.060	2.69	92	132	142	143	145
12	8832	24376	9.554	.4681	176.11	245.63	180	2.510	.812	1.52	0	4	7	21	—
13	5769	14405	5.804	.4281	173.69	165.95	180	.975	.519	2.05	—	—	—	—	—
14	4074	9053	3.776	.3792	166.98	157.65	180	1.670	1.177	8.88	0	20	—	—	—
15	NO	ORBITS													
16	5195	10476	4.508	.3370	172.71	246.38	180	1.395	0.735	4.25	0	—	—	—	—
17	3463	4639	2.331	.1452	166.28	228.40	180	0.925	0.783	12.18	—	—	—	—	—
18	5391	8677	4.047	.2335	175.59	109.66	180	1.444	1.261	8.57	10	—	—	—	—
19	r _a >	60,000													
20	4289	44284	13.974	.8234	158.78	204.71	3	0	0	0	—	—	—	—	—
21	3508	8535	3.465	.4174	173.18	261.32	180	1.455	1.103	9.47	6	—	—	—	—
22	r _a >	60,000													
23	5187	16535	6.249	.5224	157.04	192.95	180	2.334	1.135	4.02	16	18	28	—	—
24	5838	30823	10.547	.6815	162.27	171.12	180	3.588	0.654	1.06	46	47	48	49	51
25	5988	10988	4.884	.2945	175.38	40.82	180	1.653	1.252	6.42	52	126	—	—	—

TIME (DAYS) WHEN SHADOW EXCEEDS

RUN	r_p	r_a	a	e	i	ω	L	t_{max}	\bar{E}_s	$\eta \cdot t$	1h	1.5h	2h	2.5h	3h
26	$r_a > 60,000$.5513	174.37	302.18	180.	1.394	0.805	6.35	0	—	—	—	—
27	2859	9884	3.666	.6721	164.87	170.28	180	3.922	0.891	1.75	152	152	153	155	161
28	5270	26873	9.247	.3935	177.33	274.03	180	2.184	1.191	4.04	0	0	6	—	—
29	6779	15576	6.431	.4563	178.10	232.61	180	1.665	1.089	11.37	0	0	—	—	—
30	2873	7697	3.041												
31	$r_a > 60,000$														
32	$r_a > 60,000$														
33	5199	24751	8.616	.6528	165.71	171.41	180	4.135	0.881	1.93	68	144	144	145	148
34	8046	23420	9.052	.4886	170.04	212.64	180	0.524	0.063	0.13	—	—	—	—	—
35	NO ORBITS														
36	7915	27048	10.058	.5472	169.20	214.05	180	0.668	0.140	0.24	—	—	—	—	—
37	5287	43614	14.068	.7838	159.10	209.92	4	0	0	0	—	—	—	—	—
38	3823	8227	3.467	.3655	168.26	32.76	180	1.229	0.792	6.79	0	—	—	—	—
39	4252	8304	3.612	.3227	170.45	275.25	180	1.083	0.918	7.40	6	—	—	—	—
40	4164	14926	5.492	.5638	171.85	176.07	180	0.944	0.467	2.01	—	—	—	—	—
41	5211	39772	12.941	.7683	158.09	168.21	45	0.699	0.585	0.69	—	—	—	—	—
42	2823	7382	2.936	.4467	169.51	325.82	180	1.561	1.018	11.20	0	18	—	—	—
43	4001	6204	2.936	.2159	173.70	303.65	180	1.229	0.978	10.76	106	—	—	—	—
44	4115	11022	4.355	.4562	169.75	175.22	180	2.001	1.297	7.90	0	4	—	—	—
45	NO ORBITS														
46	5834	13988	5.703	.4114	175.01	160.68	180	1.054	0.539	2.19	52	—	—	—	—
47	6589	41177	13.742	.7241	163.110	160.38	46	0.802	0.660	0.72	—	—	—	—	—
48	2822	7668	3.018	.4620	170.48	147.38	180	1.642	1.187	12.53	0	34	—	—	—
49	6171	27300	9.629	.6313	169.94	207.45	180	0.757	0.264	0.49	—	—	—	—	—
50	3604	12247	4.560	.5453	169.91	28.70	180	2.239	1.219	6.93	92	124	150	—	—

TIME (DAYS) WHEN SHADOW EXCEEDS

RUN	r_p	r_a	a	e	i	ω	L	t_{max}	T_s	γ_{\odot}	1h	1.5h	2h	2.5h	3h
51	4427	6920	3.264	.2197	172.95	145.79	180	1.207	0.932	8.74	138	—	—	—	—
52	4360	14124	5.318	.5283	173.01	191.95	180	2.350	1.333	6.02	0	0	6	—	—
53	4009	45880	14.352	.8393	156.80	164.99	31	0.668	0.548	0.56	—	—	—	—	—
54	$r_a > 60,000$										—	—	—	—	—
55	5577	30155	10.280	.6879	162.30	172.21	180	0.843	0.291	0.49	—	—	—	—	—
56	NO ORBITS										—	—	—	—	—
57	5838	14122	5.742	.4150	166.21	190.57	180	2.049	1.299	5.23	4	12	32	—	—
58	2896	43132	13.242	.8741	150.81	169.99	7	0.540	0.487	0.56	—	—	—	—	—
59	5930	10930	4.850	.2965	175.39	174.24	180	1.658	1.347	6.98	0	34	—	—	—
60	4141	8082	3.516	.3224	167.80	123.60	180	1.418	1.124	9.44	6	—	—	—	—
61	3406	33360	10.577	.8147	155.20	173.41	33	0.636	0.542	0.87	—	—	—	—	—
62	4496	8477	3.732	.3068	175.35	141.19	180	1.506	1.269	9.74	0	78	—	—	—
63	5764	14903	5.946	.4422	174.06	166.32	180	1.022	0.496	1.89	66	—	—	—	—
64	3177	17419	5.925	.6915	159.52	359.83	180	2.975	1.107	4.25	14	14	16	18	—
65	2840	21063	6.877	.7624	162.18	178.06	74	0.726	0.581	1.78	—	—	—	—	—
66	5039	8706	3.954	.2668	154.36	26.12	180	0.729	0.147	1.03	—	—	—	—	—
67	4672	15823	5.896	.5441	173.01	167.36	180	0.915	0.458	1.77	—	—	—	—	—
68	3391	9666	3.756	.4806	176.78	197.45	180	1.875	1.283	9.76	0	6	—	—	—
69	4912	11585	4.746	.4045	176.05	261.90	180	1.797	1.195	6.40	0	18	—	—	—
70	$r_a > 46,300$										—	—	—	—	—
71	5295	24970	8.707	.6501	163.77	176.24	180	4.166	0.984	2.12	79	147	147	149	150
72	7453	40181	13.704	.6871	162.48	162.64	180	2.084	0.266	0.29	169	169.5	170.5	—	—
73	5058	27852	9.468	.6926	165.69	167.19	180	3.854	0.737	1.40	158	159	160	162	163
74	NO ORBITS										—	—	—	—	—
75	5415	29055	9.917	.6858	163.78	169.69	180	0.875	0.318	0.564	—	—	—	—	—

TIME (DAYS) WHEN SHADOW EXCEEDS

RUN	r_p	r_a	a	e	i	w	L	t_{smax}	\bar{t}_s	% t	1h	15h	2h	2.5h	3h
76	5026	25702	8.840	.6729	164.75	172.22	180	4.246	1.010	2.13	79	148	148	149	150
77	5538	15294	5.993	.4683	168.84	193.45	180	1.156	0.569	2.15	54	—	—	—	—
78	4619	8310	3.720	.2855	175.19	54.05	180	1.202	0.902	6.96	0	—	—	—	—
79	3140	19850	6.614	.7268	166.97	173.52	180	2.282	0.782	2.54	96	120	136	—	—
80	6085	22211	8.140	.5699	165.22	175.83	180	2.984	1.163	2.77	66	—	—	—	—
81	4692	46192	14.639	.8156	150.34	170.52	20	0.517	0.244	0.24	—	—	—	—	—
82	r_a	> 46,300													
83	3937	20679	7.082	.6801	173.41	154.40	180	1.082	0.544	1.60	110	—	—	—	—
84	4161	9374	3.894	.3852	176.45	189.18	180	1.621	1.298	9.35	4	62	—	—	—
85	7717	17683	7.307	.3924	161.28	210.47	180	1.632	0.989	2.77	28	32	—	—	—
86	3058	8841	3.423	.4860	174.31	285.70	180	1.706	1.173	10.25	0	28	—	—	—
87	4940	17713	6.517	.5639	164.80	185.03	180	1.472	0.703	2.34	80	—	—	—	—
88	5736	9405	4.356	.2423	172.91	298.40	180	1.456	1.003	6.11	0	—	—	—	—
89	4030	32996	10.652	.7823	126.04	98.43	180	1.307	0.095	1.52	97	—	—	—	—
90	3233	4956	2.356	.2103	169.81	210.14	180	0.956	0.787	12.04	—	—	—	—	—
91	4417	10852	4.393	.4214	174.07	193.08	180	1.896	1.194	7.18	0	0	—	—	—
92	5753	33079	11.171	.7037	166.11	160.93	180	0.887	0.231	0.34	—	—	—	—	—
93	r_a	> 60,000													
94	7467	44357	14.909	.7118	160.23	163.47	46	0.684	0.453	0.44	—	—	—	—	—
95	5377	21001	7.589	.5923	176.64	270.65	180	3.259	1.041	2.76	0	0	0	4	19
96	4128	32310	10.483	.7734	171.42	147.81	73	0.903	0.663	1.08	—	—	—	—	—
97	5957	9825	4.540	.2450	174.14	137.91	180	1.541	1.333	7.63	0	66	—	—	—
98	5656	33442	11.248	.7107	163.09	210.89	180	0.978	0.461	0.68	—	—	—	—	—
99	3488	8822	3.541	.4333	176.25	245.51	180	1.626	1.236	10.27	0	48	—	—	—
100	5079	20300	7.301	.5998	168.54	172.52	180	1.864	0.754	2.12	79	109	—	—	—

FIGURE 1

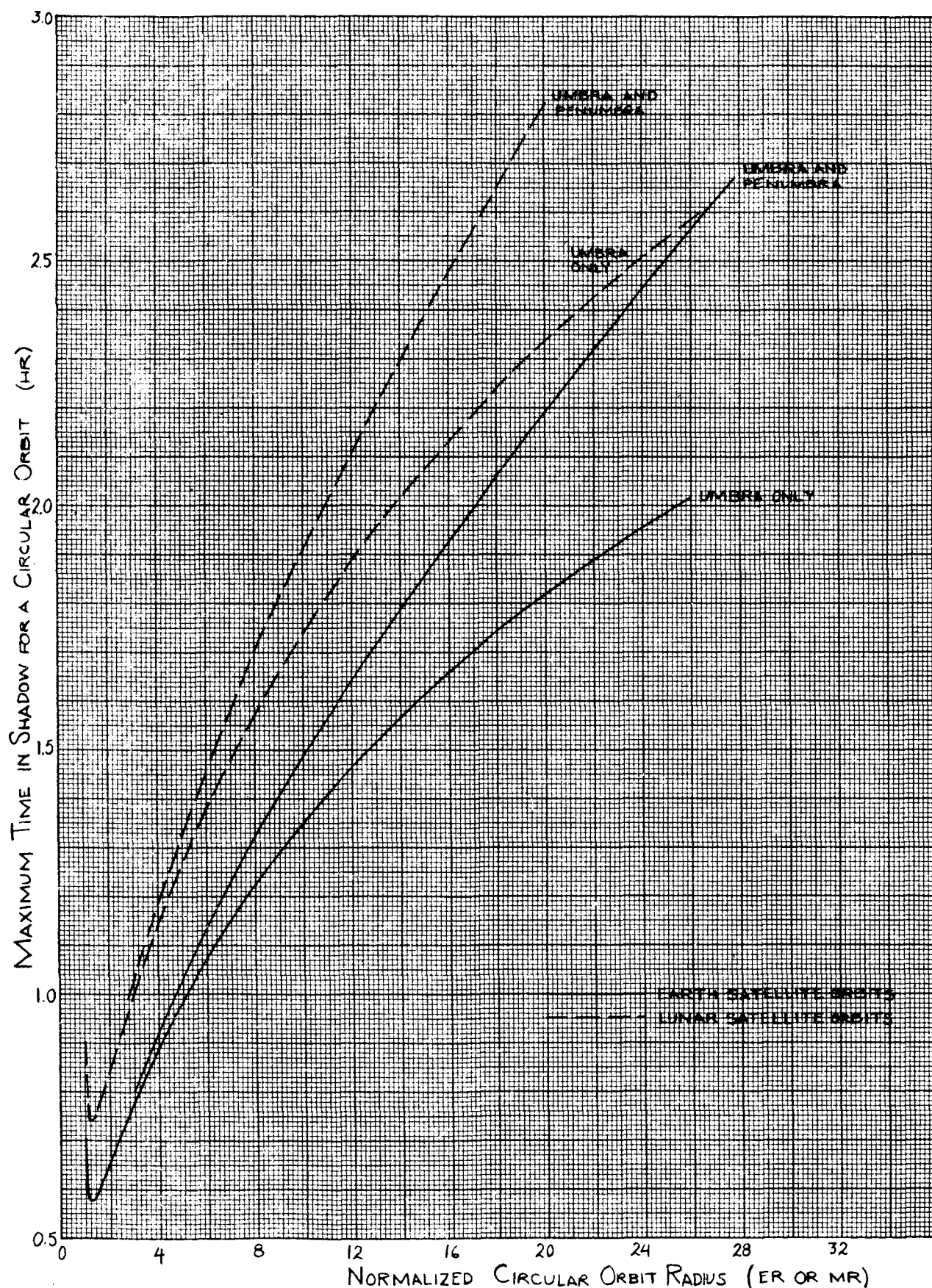
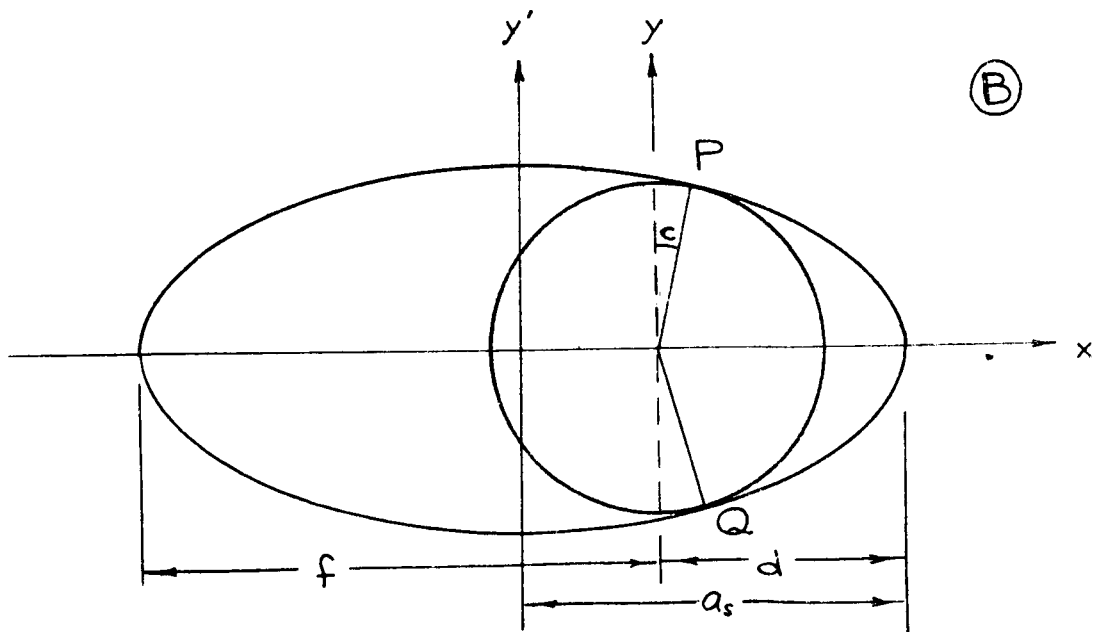
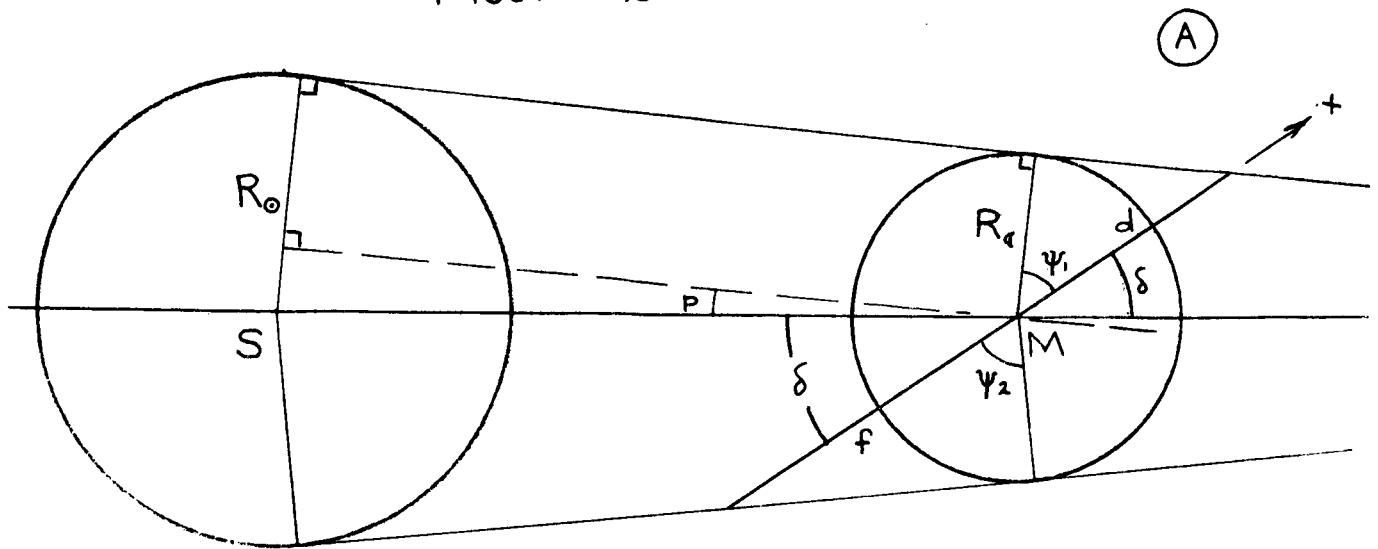


FIGURE 2



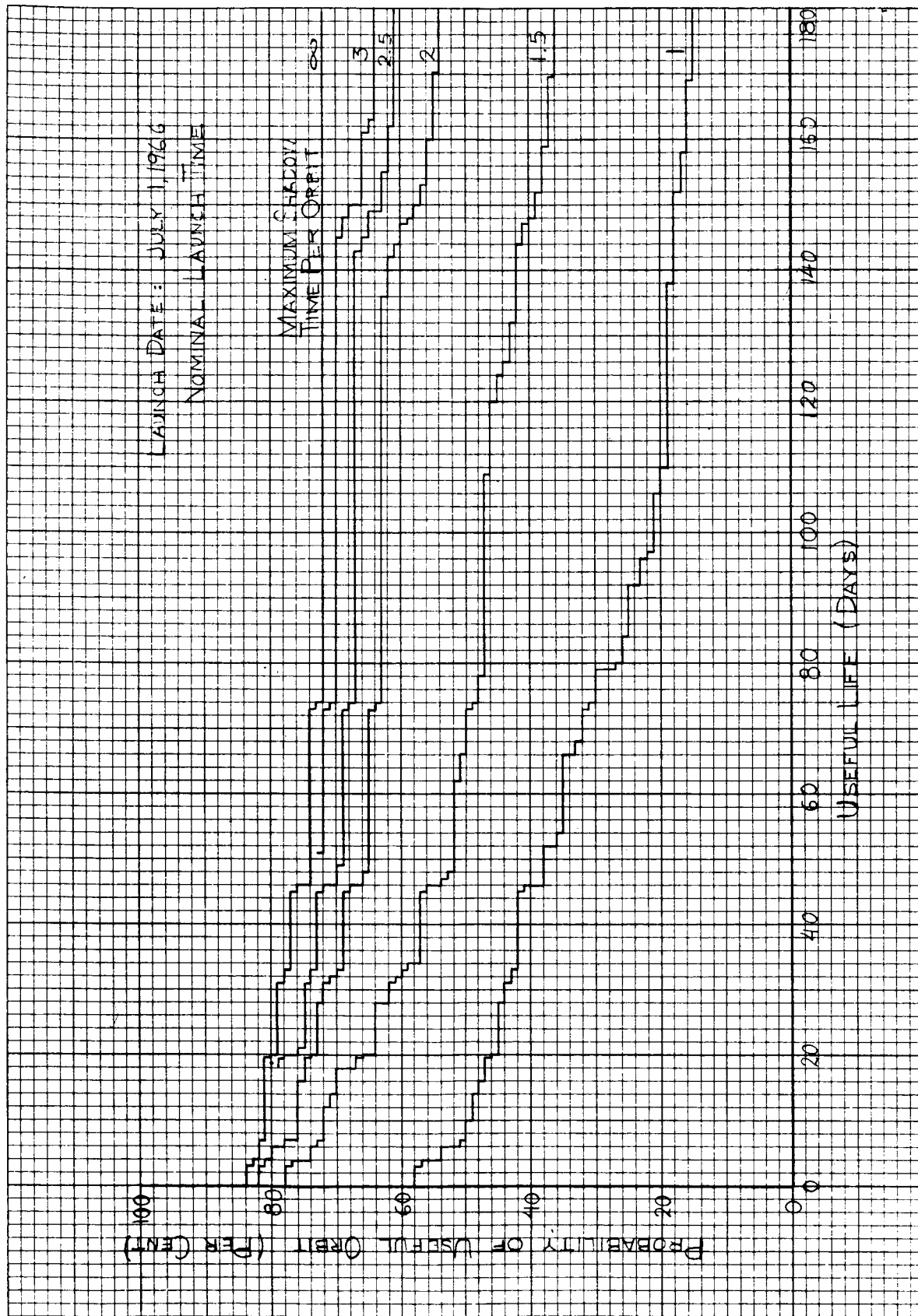


FIGURE 3

FIGURE 4

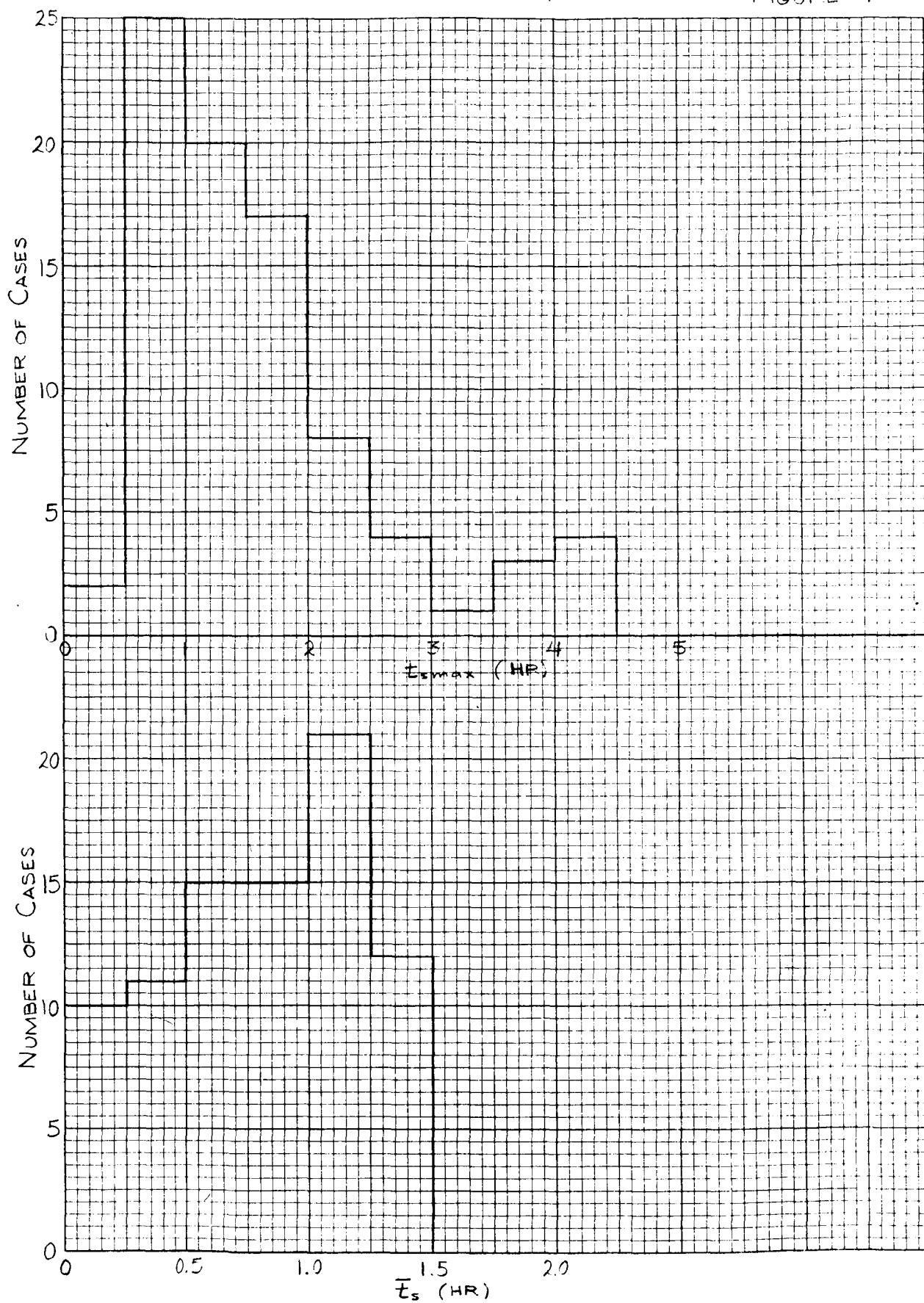


FIGURE A-1
VARIATION IN UMBRA AND PENUMBRA
HALF CONE ANGLES FOR 1966

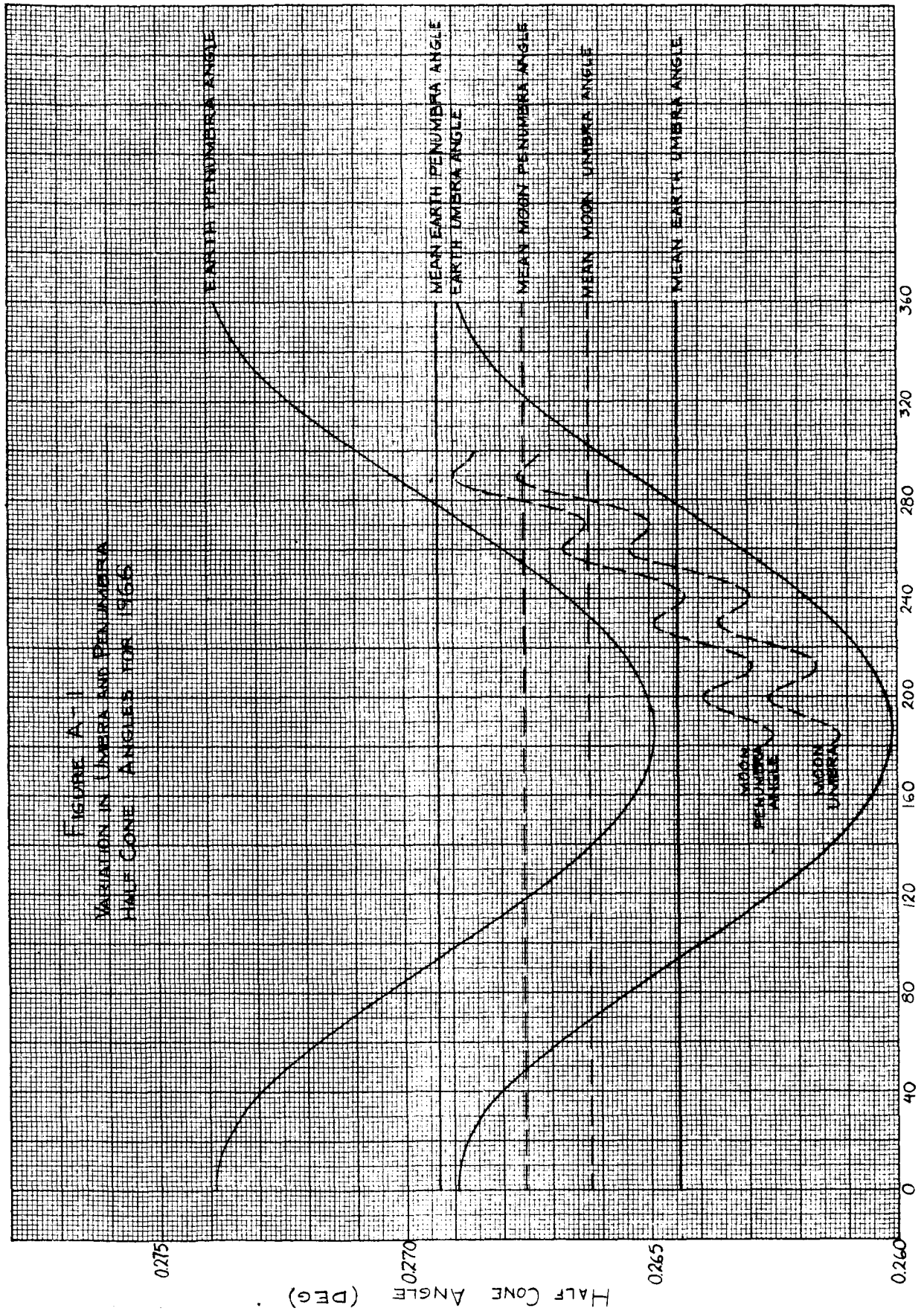
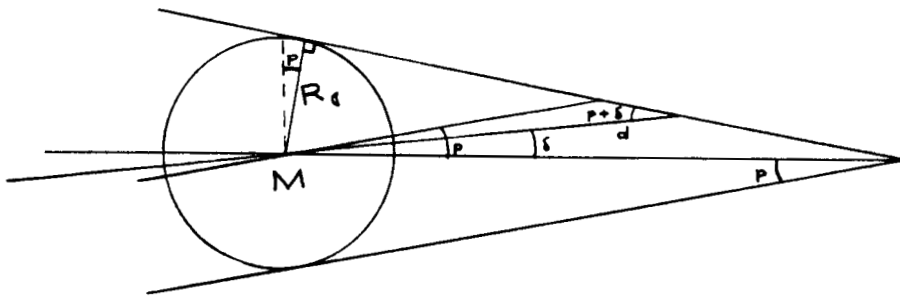


FIGURE B-1

(A)



(B)

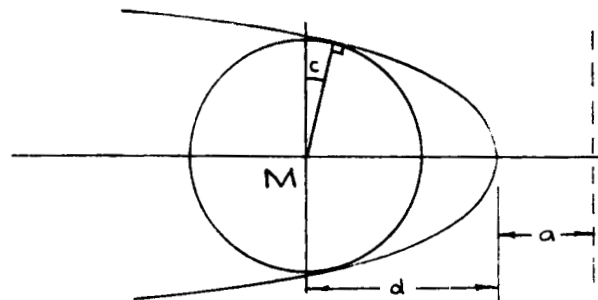
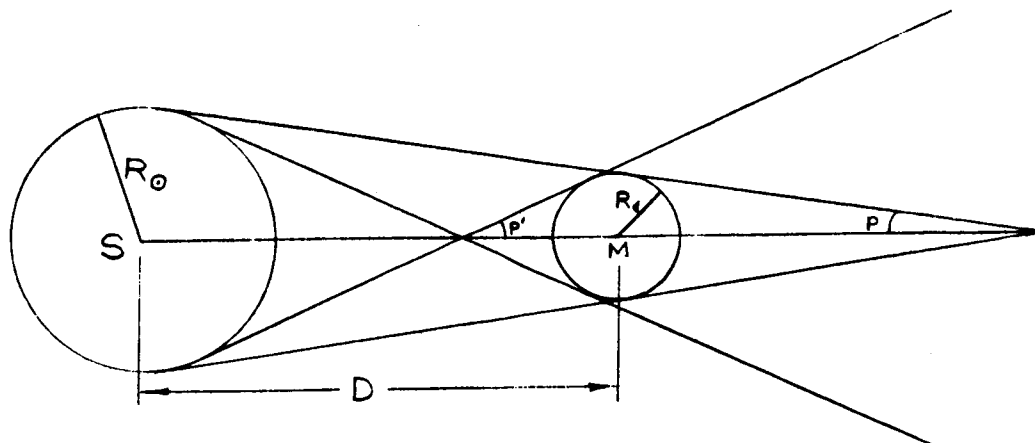
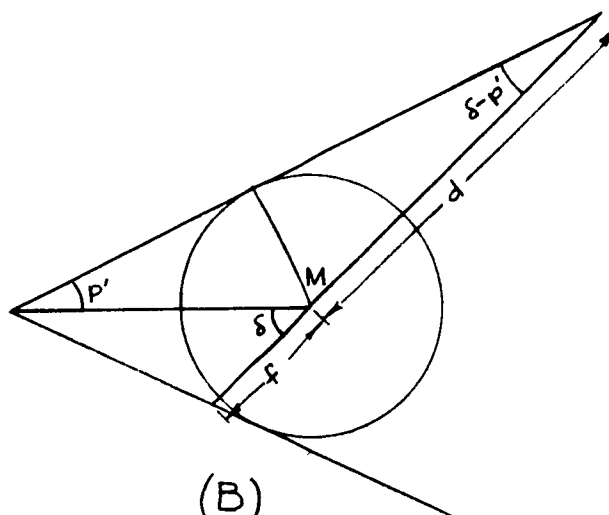


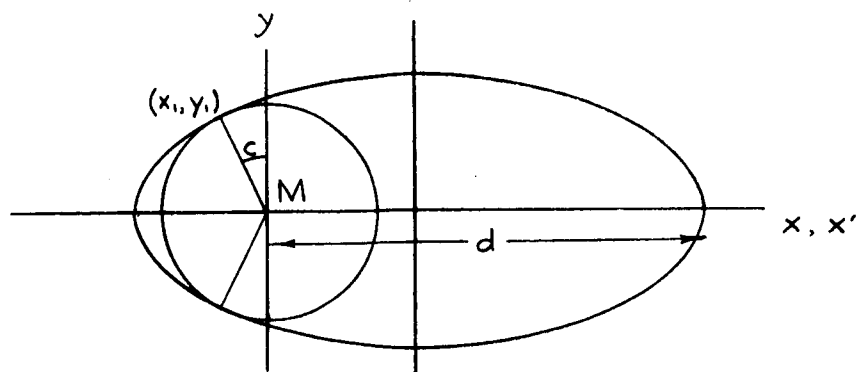
FIGURE C-1



(A)



(B)



(C)